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Approximate Transfer Functions for Large Aspect Ratio Wings in Turbulent Flow

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The fundamental aerodynamic problem associated with flight through atmospheric turbulence is calculation of the response of a wing flying through a single sinusoidal wave of upwash with lines of constant phase arbitrarily inclined to the flight path. Approximate closed form expressions for the "gust transfer functions" relating the lift and moments to the upwash in such a wave are derived for large aspect ratio rectangular wings in incompressible flow. The lift transfer function is expressed as the usual two-dimensional Sears function multiplied by a factor to correct for finite span and a further factor to account for the spanwise gust wave number. Multiplying this expression by the chordwise (or spanwise) center of pressure leads to the pitching (or rolling) moment transfer function. Some simple scaling laws, based on these results, are then suggested for flight through large-scale turbulence.

Nomenclature

A = aspect ratio = s/b
b = wing semichord
$C(k)$ = Theodorsen function = $H_1^{(2)}(k)/[H_1^{(2)}(k) + iH_0^{(2)}(k)]$
$\hat{C}(k,y)$ = local Theodorsen function, see Eq. (3)
C_L = wing lift coefficient = $L/(2\rho U^2 bs)$
C_P = wing pitching moment coefficient about $\frac{1}{4}$ chord
C_R = wing rolling moment coefficient
$C_l(y)$ = section lift coefficient
$E(y)$ = envelope function, see Eq. (31)
$F(x)$ = Cicala function, see Ref. 13
$G(z)$ = $K_1(z) - i\{1 - \pi/2[L_1(z) - I_1(z)]\}$
$J(k)$ = $J_1(k)/[J_0(k) - iJ_1(k)]$
k = reduced frequency = $b\omega/U$
$\hat{k}_1$ = streamwise wave number, see Fig. 1
$\hat{k}_2$ = spanwise wave number, see Fig. 1
k_1 = $b\hat{k}_1$
k_2 = $s\hat{k}_2$
L = total wing lift
p = local pressure jump across wing
$R(k)$ = $C(k) + iJ(k)$
$S(k_1)$ = Sears function = $(2/\pi k)[H_0^{(2)}(k) - iH_1^{(2)}(k)]^{-1}$
$S_G(k_1)$ = generalized Sears function, see Eq. (16)
s = wing semispan
$\hat{t}$ = time
t = $U\hat{t}/b$
U = flight speed
w = instantaneous vertical velocity
$\bar{w}$ = magnitude of w
$\hat{x}$ = streamwise coordinate, origin at midchord
x = $\hat{x}/b$
$\hat{y}$ = spanwise coordinate, origin at midspan
y = $\hat{y}/s$
$\hat{x}_{cp}$ = distance between center of pressure and leading edge
$I_n, J_n, K_n, L_n, H_n^{(2)}$ = Bessel functions in usual notation, see Ref. 12
α_0 = instantaneous angle of attack at midwing
ϕ_L = one-dimensional lift spectrum, see Eq. (20)
ϕ_{WW} = two-dimensional turbulence spectrum
ϕ_W = one-dimensional turbulence spectrum, see Eq. (22)
ϕ_{W_e} = "effective" turbulence spectrum, see Eq. (24)
$\mu(k)$ = $(\pi k)^{-1}[J_0(k) - iJ_1(k)][H_0^{(2)}(k) - iH_i^{(2)}(k)]$
Γ = circulation normalized by two-dimensional value
$\hat{\Gamma}$ = average normalized circulation, see Eq. (7)
ρ = fluid density
ω = circular frequency
Λ = turbulence integral scale

Subscripts

P = pitching moment/$(4\rho U^2 b^2 s)$
R = rolling moment/$(4\rho U^2 bs^2)$

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Introduction

IN recent years generalized harmonic analysis has become increasingly entrenched as the fundamental tool for calculation of the dynamic response of aircraft to atmospherie turbulence. In general, a two-dimensional theory is needed to account for instantaneous spanwise variations of upwash. Of two alternate formulations, the spectral approach of Ribner[1] as developed by Etkin[2,3] is conceptually simpler (although fundamentally equivalent to) the correlation approach originally suggested by Liepmann[4,5] and later extended by Diederich.[6,7]

In the spectral approach the fundamental aerodynamic problem is determination of the response to a single spectral (or Fourier) component of turbulence. Such a component is visualized[1] as a sinusoidal gust of arbitrary wavelength with lines of constant phase inclined at an arbitrary angle to the flight path (Fig. 1). The forces and moments acting

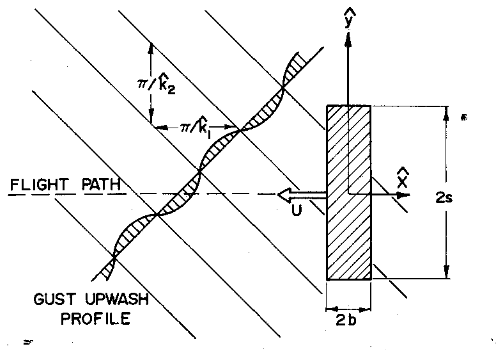


Fig. 1 Basic aerodynamic problem associated with response to atmospheric turbulence: flight through an oblique sinusoidal gust.

on an aircraft flying a rectilinear constant speed path through such an "oblique sinusoidal gust" constitute aerodynamic transfer functions (or frequency response vectors) which appear as forcing terms in the equations of motion of the unrestrained airplane.[2,3]

In general, direct computation of the transfer functions for a particular planform and Mach number is unwieldy (if indeed not intractable). If the turbulence integral scale is at least twice the span, the transfer functions can be approximated by summations of simpler responses.[1,2] It is noteworthy that the transfer functions can also be expressed in terms of simpler responses for the general case through use of reverse flow methods. Considerable simplification in the aerodynamic calculations may result; alternately, increased accuracy is possible for a given amount of labor.

An approximate expression for the lift transfer function for large aspect ratio unswept wings in incompressible flow is derived on the basis of this idea. The argument proceeds as follows: for wings of large aspect ratio the integral equation of lifting surface theory can be simplified as in any of several successful "lifting line" type theories. In particular, the theory of Reissner[8,9] agrees with experimental results for aspect ratios as low as two for certain simple modes of oscillation.[10] This theory is not, however, directly applicable to the case illustrated by Fig. 1 if the spanwise projection of the gust wavelength is small; for in this instance, the fundamental assumption of weak spanwise gradients is violated. An indirect approach using the Reissner formulation is nevertheless possible.

The reverse flow theorem of Flax[11] shows that the pressure distributions on symmetric planforms executing certain simple oscillations form influence functions for the total lift or moments in an arbitrary upwash. The Reissner theory often adequately describes these simpler pressure distributions. In particular, the pressure distribution for the lift influence function can be expressed (approximately) in closed form. The total lift during flight through the sinusoidal gust of Fig. 1 then follows. Transfer functions for the pitching and rolling moments are then suggested from heuristic arguments. The present results will offer no relief from the laborious numerical calculations necessary for analysis of the gust response of a particular configuration; rather they are intended to provide physical insight into the fundamental aerodynamics and a basis for comparison of various approximations.

Response to Plunging Oscillations: Lift Influence Function

The vertical velocity distribution over a plane wing executing harmonic plunging oscillations may be written

$$w = \bar{w}e^{\pm ikt} \tag{1}$$

Consider a rectangular wing of high aspect ratio. If the span were infinite the pressure distribution would be given by two-dimensional theory (e.g., Ref. 10) as

$$p = 2\rho Uw\{C(k)[(1-x)/(1+x)]^{1/2} + ik(1-x^2)^{1/2}\} \tag{2}$$

For finite (but large) aspect ratio the theory of Reissner[8,9] accounts for three-dimensionality of the flow by replacing the Theodorsen function $C(k)$ in Eq. (2) by a modified function of spanwise position $\hat{C}$, which can be written as

$$\hat{C}(k,y) = R(k)\Gamma(y) - iJ(k) \tag{3}$$

Here R and J are combinations of Bessel functions[12] defined in the list of symbols. The function $\Gamma(y)$ is the ratio of the actual circulation around a spanwise section to the value which would be calculated using two-dimensional theory. For the rectangular wing Γ is obtained from an integral equation which in the present notation has the form

$$\Gamma(y) + \frac{\mu(k)}{A}\int_{-1}^{1}\frac{d\Gamma(\eta)}{d\eta}\left\{\frac{1}{y-\eta} - iAkF[Ak(y-\eta)]\right\}d\eta = 1 \tag{4}$$

The function μ is another combination of Bessel functions (see Nomenclature) as is the Cicala function F.[13]

Equation (4) usually requires numerical solution: however, Reissner and Stevens point out[9] that for the simple velocity function [Eq. (1)] sufficient accuracy is obtained assuming the form

$$\Gamma(y) \sim (1-y^2)^{1/2} \tag{5}$$

The integrated value

$$\hat{\Gamma}(y) = \tfrac{1}{2}\int_{-1}^{1}\Gamma(y)dy \tag{6}$$

may also be expressed in closed form to a good approximation by "averaging" Eq. (4) over the span.[14] The result is

$$\hat{\Gamma}(k,A) = [1 + 4k\mu(k)G(kA)]^{-1} \tag{7}$$

where G is yet another combination of Bessel functions (see Nomenclature). Equation (7) may be regarded as an interpolation formula between the lifting line result for zero frequency and the two-dimensional result for very large frequency.

From Eqs. (5) and (7) we then find

$$\Gamma(y) = (4/\pi)\hat{\Gamma}(k,A)(1-y^2)^{1/2} \tag{8}$$

Actually, this simple elliptic spanwise shape cannot be correct for very large values of the reduced frequency parameter k: it may, in fact, be shown from Eq. (4) that the high-frequency loading is uniform across the span to the order k^{-2}. The total response, however, falls off rapidly as the frequency increases; since the integrated value is correct to the order k^{-2}, the assumed shape [Eq. (5)] does not introduce significant error in the final answer.

The pressure distribution over the large aspect ratio rectangular wing with velocity distribution [Eq. (1)] is obtained using Eq. (8) in Eq. (3) and substituting for C in Eq. (2):

$$p(x,y,t) = 2\rho Uw\{4/\pi R(k)\hat{\Gamma}(k,A)[(1-x)/(1+x)]^{1/2} \times (1-y^2)^{1/2} - i[J(k)(1-x)^{1/2}(1+x)^{-1/2} - k(1-x^2)^{1/2}]\} \tag{9}$$

The pressure distribution corresponding to uniform flight at constant angle of attack $\bar{w}/U$ is obtained by setting $k = 0$ in Eq. (9) as

$$p(x,y) = 8/\pi\rho U\bar{w}A(A+2)^{-1}[(1-x)/(1+x)]^{1/2} \times (1-y^2)^{1/2} \tag{10}$$

To the present approximation, then, the uniform flow chordwise loading is of the classical flat-plate type and the spanwise loading is elliptic.

Response to a Single Spectral Component: Lift Transfer Function

The wing flying through a single spectral component of turbulence (Fig. 1) sees an oblique sinusoidal wave pattern of upwash

$$w_G(x,y,t) = e^{i[\hat{k}_1(\hat{x} - U\hat{t}) + \hat{k}_2\hat{y}]} = e^{i[k_1x + k_2y]}e^{-ik_1t} \tag{11}$$

Since the time dependence is harmonic (circular frequency $\omega = k_1U/b$) the total lift can be expressed in terms of the distribution of lift associated with the simpler velocity field Eq. (1) by means of the reverse flow theorem (more accurately termed a reciprocity relation) of Flax[11]

$$\int p_D w_R dS = \int p_R w_D dS \tag{12}$$

Here the subscripts D and R refer to harmonic motions at the same frequency in direct and reverse flows. The integrals are taken over the wing planform.

Let the reverse flow refer to harmonic plunging of frequency k_1U/b and unit amplitude. For rectangular planform the reverse flow pressure distribution is simply obtained by changing the sign of x wherever it occurs in Eq. (9). Thus Eq. (12) may then be used to calculate the total lift in the direct flow:

$$L(t) = 2\rho Ub^2A \int_{-1}^{1} dx \int_{-1}^{1} dy\{4/(\pi A)R(k_1)\Gamma(k_1,A) \times [(1-x)/(1+x)]^{1/2}(1-y^2)^{1/2} - i[J(k_1) \times (1-x)^{1/2}(1+x)^{-1/2} - k_1(1-x^2)^{1/2}]\}\, w_D(x,y,t) \tag{13}$$

This expression is quite general giving the total lift in an arbitrary (harmonic) velocity field: the restriction is that the Reissner theory should apply for simple plunging oscillations:

Using Eq. (11) for the direct flow velocity, the integrations can be carried through with the result

$$C_{LG} = L_G(2\rho U^2bs)^{-1} = 2\pi\alpha_0 S_G(k_1,k_2,A) \tag{14}$$

where α_0 is the instantaneous angle of attack at midwing

$$\alpha_0 = U^{-1}e^{-ik_1t} \tag{15}$$

and S_G is a "generalized Sears function" or frequency response function:

$$S_G(k_1,k_2,A) = S(k_1)\hat{\Gamma}(k_1,A)2J_1(k_2)/k_2 \tag{16}$$

The factor

$$S(k_1) = S_G(k_1,0,\infty)$$

is the usual two-dimensional sinusoidal gust response function of Sears.[15] An explicit representation in terms of Bessel functions is given in the Nomenclature. The factor $\hat{\Gamma}$ accounts for finite aspect ratio. Finally, the remaining factor $2J_1(k_2)/k_2$ accounts for spanwise variations in gust intensity.

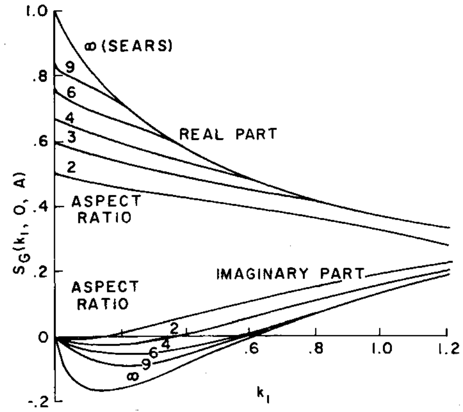


Fig. 3 Effect of aspect ratio on lift transfer function in normal gust.

One-Dimensional Functions

If the turbulence integral scale is much greater than the span $2bA$ "one-dimensional" or lifting point analysis suffices.[1,4] In this instance the transfer function corresponding to flight through an uninclined gust (Fig. 2), is obtained by setting k_2 equal to zero in Eq. (16). The result

$$S_G(k,0,A) = S(k)\hat{\Gamma}(k,A) \tag{17}$$

which was obtained in a different way in Ref. 14, is indicated in Fig. 3. Equation (17) interpolates between the steady-state lifting line result $S_G = A/(A+2)$ and the two-dimensional result which applies for large k. The finite span effect persists to larger values of kA in the out-of-phase part than in the in-phase part.

Another special case may be termed "one-dimensional" in a certain sense. This is the case $k_1 = 0$, corresponding to flight parallel to the gust nodal lines (Fig. 4), which can be interpreted as steady flight of a sinusoidally twisted wing. For this case

$$\left.\frac{C_L(k_2)}{C_L(0)}\right|_{k_1=0} = 2\,\frac{J_1(k_2)}{k_2} \tag{18}$$

This is the exact lifting surface theory result for any symmetric wing supporting elliptic loading in steady flight at uniform angle of attack (as may be demonstrated by direct application of the steady version of the reverse flow theorem).

Spectral Relations

These results may now be applied to flight through atmospheric turbulence. The mean square lift coefficient can be expressed as the integral of a spectrum function

$$\langle C_L^2\rangle = \int_0^\infty \phi_L(\omega)d\omega \tag{19}$$

Here $\omega = k_1U/b$ is a circular frequency. In the present nota-

Fig. 2 Normal gust case: $k_2 = 0$.

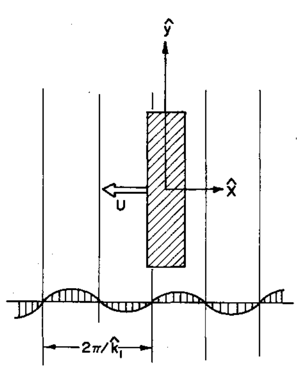
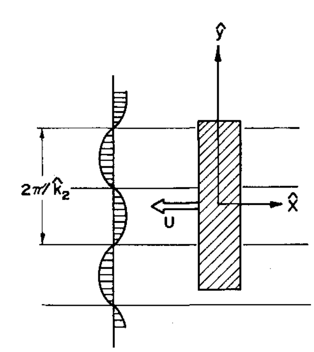


Fig. 4 Transverse gust case: $k_1 = 0$.



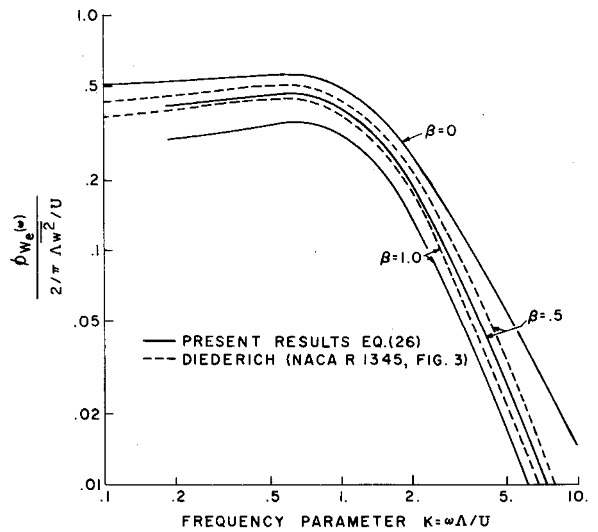


Fig. 5 Normalized "effective" turbulence spectrum.

tion the spectrum function is written

$$\phi_L(\omega) = 2/U \int_{-\infty}^{\infty} |2\pi U^{-1} S_G(b\omega U^{-1},k_2,A)|^2 \times \phi_{WW}(\omega U^{-1},k_2 b^{-1} A^{-1}) dk_2 \quad (20)$$

The mean square vertical velocity fluctuation is

$$\langle w^2 \rangle = \int\int_{-\infty}^{\infty} \phi_{WW}(\hat{k}_1,\hat{k}_2) d\hat{k}_1 d\hat{k}_2 \quad (21)$$

Now if the turbulence were one-dimensional the lift coefficient spectrum could be expressed in terms of a one-dimensional velocity spectrum[1,2]

$$\phi_W(\hat{k}_1) = \int_{-\infty}^{\infty} \phi_{WW}(\hat{k}_1,\hat{k}_2) d\hat{k}_2 \quad (22)$$

as

$$\phi_L(\omega)|_{1-\text{dim}} = |2\pi U^{-1} S_G(\omega b U^{-1},0,A)|^2 \phi_W(\omega) \quad (23)$$

By analogy with this expression Diederich[6,7] writes the spectral relationship in two-dimensional turbulence in terms of an "effective" one-dimensional velocity spectrum ϕ_{W_e} as

$$\phi_L(\omega) = |2\pi U^{-1} S_G(\omega b U^{-1},0,A)|^2 \phi_{W_e}(\omega) \quad (24)$$

Comparison of Eqs. (20) and (24) with use of Eq. (16) for S_G gives

$$\phi_{W_e}(\omega) = 8(UbA)^{-1} \int_{-\infty}^{\infty} [J_1(k_2)/k_2]^2 \phi_{WW}(\omega U^{-1},k_2 b^{-1} U^{-1}) dk_2 \quad (25)$$

For explicit results the turbulence spectrum has to be specified. A convenient analytical form[2-6] is the Dryden spectrum

$$\phi_{WW}(\hat{k}_1,\hat{k}_2) = 3/(4\pi)\Lambda^4 \langle w^2 \rangle (k_1^2 + k_2^2)[1 + \Lambda^2(k_1^2 + k_2^2)]^{-5/2} \quad (26)$$

With this expression Eq. (25) reduces to

$$\frac{\phi_{W_e}(\omega)}{2\Lambda\langle w^2 \rangle (U\pi)^{-1}} = 6\beta^2 \int_0^{\infty} \frac{J_1^2(x)}{x^2} \frac{\beta^2 K^2 + x^2}{[\beta^2(1 + K^2) + x^2]^{5/2}} dx \quad (27)$$

where

$$\beta = s/\Lambda, \; K = \omega\Lambda/U$$

The integral in Eq. (27) may be expressed in closed form[16] as the sum of four hypergeometric functions of the type ${}_3F_4$; however, for present purposes numerical evaluation was found more convenient. The left-hand member of Eq. (27) was calculated by Diederich[6] in a completely different way. His results are compared with the present analysis on Fig. 5. The two sets of curves are similar in shape with the present result showing a slightly greater influence due to finite span, perhaps attributable to the assumed rectangular steady-state lift distribution inherent in the curves of Ref. 6.

Pressure Distribution

As the wing passes through the inclined gust (Fig. 1), the lines of constant upwash translate along the span with uniform velocity $U \tan(\hat{k}_1/\hat{k}_2)$. If the aspect ratio is large the loading over inboard sections will be only weakly influenced by the tips. This is not to imply that strip theory is applicable, but rather that near midspan the pressure distribution will resemble that on an infinite span wing (Fig. 6).

The instantaneous spanwise loading supported by such a wing flying through an oblique sinusoidal gust is known[17] to be sinusoidal with wavelength $2\pi/k_2$ matching the spanwise projection of the gust: the nodal lines of the pressure distribution are perpendicular to the leading edge. As time progresses, the pressure wave is swept along the span maintaining a constant phase shift relative to the velocity wave (Fig. 6).

It seems reasonable to suppose that finite span will modify this simple moving sinusoidal pattern through superposition of an envelope to ensure the instantaneous vanishing of lift at the tips. In general, the shape of this envelope will depend on the gust wavelength and inclination. An aspect ratio dependent phase shift may also exist.

Chordwise Loading: Pitching Moment Transfer Function

If, as assumed, finite aspect ratio does not significantly alter the shape of the chordwise loading, we can make use of the infinite span result[17]

$$p \sim [(1 - x)/(1 + x)]^{1/2} e^{-\hat{k}_2 \hat{x}} \quad (28)$$

This shape is very similar to the chordwise loading carried by a finite wing of aspect ratio $(b\hat{k}_2)^{-1}$ in steady flight at constant angle of attack.[17] That is, the instantaneous chordwise loading is similar in shape to the steady-state chordwise loading on a finite wing with aspect ratio equal to that of the portion cut out of the infinite wing by two successive nodal lines of the gust.

The center of pressure corresponding to Eq. (28) is located a distance

$$\hat{x}_{cp} = (1/\hat{k}_2) I_1(\hat{k}_2 b)/[I_0(\hat{k}_2 b) + I_1(\hat{k}_2 b)] \quad (29)$$

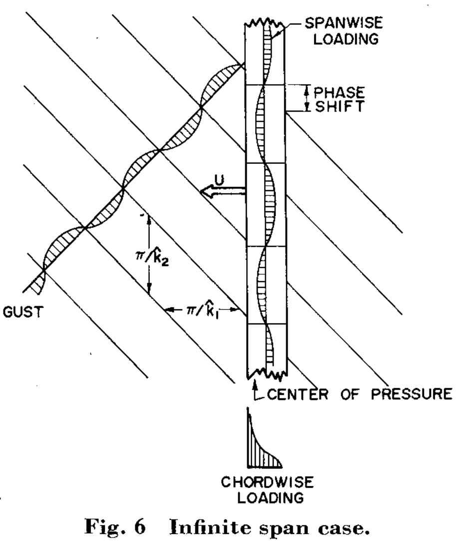


Fig. 6 Infinite span case.

aft of the leading edge. Accordingly, the pitching moment coefficient about the quarter chord line is

$$C_p = 1/2b[(b/2) - \hat{x}_{cp}]C_L = \pi/2\{1 - 2/(k_2/A)I_1(k_2/A)/[I_0(k_2/A) + I_1(k_2/A)]\}S_G\alpha_0 \quad (30)$$

This formula indicates that for the normal gust case ($k_2 = 0$, Fig. 2) the lift acts through the quarter chord line for all (large) aspect ratios. As k_2 increases, the center of pressure moves forward: the resulting pitching moment is always in phase with the lift but eventually decreases with a further increase in k_2 due to the fall-off in total lift.

Spanwise Loading: Rolling Moment Transfer Function

According to the foregoing concepts, the local lift coefficient should have the form of a modulated traveling wave

$$C_l(y) = 2\pi U^{-1}E(y)e^{i(k_2y - k_1t)} \quad (31)$$

where the "envelope function" E is understood to depend implicitly on the gust parameters and the wing aspect ratio.

To satisfy the condition of zero lift at the tips

$$E(\pm 1) = 0$$

For consistency with Eq. (10)

$$\lim_{\substack{k_1\to 0\\ k_2\to 0}} E(y) = (4/\pi)A(A + 2)^{-1}(1 - y^2)^{1/2} \quad (32)$$

It is expected that the "shoulders" of the envelope become progressively squarer as the gust wavelength decreases. That is, the influence of a tip on a particular spanwise station should decrease with increasing distance between the tip and that station—whether this distance is expressed in terms of chord lengths or wavelengths.

It is possible to contrive expressions for the envelope function which simultaneously satisfy the previous conditions, integrate to the correct total lift [Eq. (14)], and have the correct limiting form[17] for infinite span. These expressions, however, are complex and of undeterminable accuracy. A very simple result, valid for small k_2, is obtained by assuming that the envelope remains elliptic and requiring the total lift to agree with Eq. (14). This leads to

$$E(y) = (4/\pi)S(k_1)\hat{\Gamma}(k_1,A)(1 - y^2)^{1/2} \quad (33)$$

$$C_R = i[J_2(k_2)/J_1(k_2)]C_L \quad (34)$$

The quantity i multiplying the right-hand side of Eq. (34) signifies a 90° phase shift between the lift and rolling moment. This phase shift exists because symmetrical gust components contribute to the lift but not to the rolling moment and vice-versa for asymmetric components. The assumption of elliptic envelope leading to Eq. (34) is expected to decrease in accuracy with increasing k_2. Thus Eq. (34) is to be used in the form

$$C_R \simeq \tfrac{1}{4}ik_2C_L, \qquad k_2 \to 0 \quad (35)$$

Response to Large-Scale Isotropic Turbulence

To estimate the root-mean-square responses during flight through turbulence consider a single spectral component "most typical" of the turbulence. In isotropic turbulence a typical chordwise or spanwise wave number will be inversely proportional to the integral scale Λ. That is

$$\hat{k}_1 = k_1b^{-1} \sim \Lambda^{-1}$$
$$\hat{k}_2 = k_2s^{-1} \sim \Lambda^{-1} \quad (36)$$

If the scale is much greater than the span, expansion of the preceding results yields the estimates

$$\left.\begin{aligned}[\langle C_L{}^2\rangle]^{1/2} &\simeq 2\pi[\langle\alpha^2\rangle]^{1/2}A(A + 2)^{-1}\\ [\langle C_p{}^2\rangle]^{1/2} &\sim [\langle\alpha^2\rangle]^{1/2}A(A + 2)^{-1}(\Lambda/b)^{-2}\\ [\langle C_R{}^2\rangle]^{1/2} &\sim [\langle\alpha^2\rangle]^{1/2}A(A + 2)^{-1}(\Lambda/s)^{-1}\end{aligned}\right\} \Lambda \gg s \gg b \quad (37)$$

The root-mean-square lift coefficient is (as expected in large-scale turbulence) given by quasi-steady considerations. For a given wing the mean square pitching moment falls off more rapidly with increasing turbulence scale than does the mean square rolling moment.

Concluding Remarks

The foregoing analysis was restricted to large aspect ratio rectangular wings in incompressible flow. The restriction on planform is not severe; consistent with the approximate nature of the derivation of the transfer functions, these should serve for any large aspect ratio wing of small sweep if the normalized wave number k_1 is understood to refer to the mean chord.

Often, more refined calculations (e.g. Ref. 18) are available for the normal gust case, Fig. 2. These may be modified to include the effect of spanwise wave number by multiplication by the factor $2J_1(k_2)/k_2$ [Eq. (16)]. This more accurate expression for the lift could then be used in conjunction with Eqs. (30) and (34) to obtain the corresponding pitching and rolling moments.

Compressibility effects are difficult to assess in a simple way. If, however, the combination of frequency and Mach number is such that terms of order $(k_1M)^2/(1 - M^2)$ are negligible, the similarity rule of Miles[19] can be used to correct Eq. (9) for compressibility; the remainder of the analysis would proceed as outlined.

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Application of Finite-Element Theory to Airplane Configurations

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Some applications of numerical lifting-surface theory for the calculation of steady subsonic and supersonic flow over complete airplane configurations are presented. Major emphasis is placed on the evaluation of a numerical finite-element theory as an aerodynamic loads prediction method for complex airplane geometries. Experimental pressure distribution data are compared with the distributed-singularity solutions in order to demonstrate the applicability as well as some of the limitations of the linearized theory for actual airplanes. The results reveal that the numerical theory can be useful for aerodynamic evaluation of complex airplane configurations. Proper usage of the theory, however, does require an understanding of the aerodynamics of the configuration and the limitations of the numerical method.

Nomenclature

b = wing span
c = wing chord
C_L = lift coefficient
C_M = pitching moment coefficient
C_p = pressure coefficient
l = body length
M = Mach number
x = chordwise distance
y = spanwise distance
α = angle of attack
Λ_{LE} = leading-edge sweep angle
θ = body meridian angle measured clockwise from vertical

Introduction

Aerodynamic design and evaluation has relied for many years on classic aerodynamic theories for thin wings and slender bodies, salted heavily with empirical factors, and on extensive wind-tunnel testing. In recent years, however, the appearance of highly efficient digital computers has opened the door for renewed emphasis on development of sophisticated analytical aerodynamic theories. The extent of this renewed emphasis on numerical methods was perhaps best illustrated at a recent NASA conference on analytical methods in aircraft aerodynamics,[1] where the development and results of numerous sophisticated computer programs were featured.

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The authors have evaluated several numerical lifting surface theories for a variety of wing configurations by comparison of theoretical results with experimental data.[2] The numerical methods include both collocation and finite-element theories, which were compared with each other, where applicable, and with experimental data. These comparisons reveal that the several numerical methods based on linearized theory give essentially equivalent results for wing-alone problems.

While wing design is an important part of the design process, the airplane designer is still faced with a multitude of problems which cannot be treated within the framework of wing-alone theory. Interfering flowfield effects are of paramount importance for practical airplane design, where the acid test for theory is agreement with wind-tunnel test. Wing-body interaction and blending, flowfield specification at inlet locations, rigid aerodynamic loading on components, and nacelle or store effects on the vehicle are some problems of concern. One analytical tool for attacking these problems is finite-element theory. Large computing facilities in common usage today make the distributed-singularity approach (and the associated-matrix-inversion requirements) feasible for complex configurations.

Application of theory to practical airplane design must be predicated on a basic understanding of the limitations of the theory and on an experimentally confirmed level of confidence. It is the purpose of this paper to examine one of the currently available finite-element methods in the context of aerodynamic-loads prediction for real airplanes.

Theory

The finite-element theory of Woodward[3] forms the basis for the following applications. The numerical method is an extension by Woodward and Hague[4] of a supersonic wing-